FOURIER SERIES AND PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS

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ABSTRACT: We use Fourier series to find a necessary and sufficient condition for existence and uniqueness of periodic solutions of differential equations on Hilbert spaces.
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1. INTRODUCTION

Let E be a Banach space and $f(t)$ be a continuous function from $[0,T]$ to E. The Fourier coefficient of $f(t)$ is defined as

$$f_k = \frac{1}{T}\int_0^T f(s)e^{-2k\pi is/T}ds,\ \ k\in\mathbb{Z}.$$

Then $f(t)$ can be represented by Fourier series

$$f(t)\sim\sum_{k=-\infty}^{\infty} e^{2k\pi it/T}f_k.$$

Using Fourier series is a well known method for investigating solutions of differential equations, in particular for periodic and almost periodic solutions (see e. g. [1], [4], [7], [9], [10]). In this paper we systematically use Fourier series to find the conditions for existence and uniqueness of periodic solutions of functional differential equation

$$x'(t) = \int_0^\infty [dE(s)]x(t-s) + f(t),$$

of the first order differential equation

$$x'(t) = Ax(t) + f(t),$$

and of the second order differential equation

$$x''(t) = Ax(t) + f(t)$$

on Hilbert spaces. The paper is organized as the follows: First we find, for any function $f \in C([0,1], E)$, a necessary and sufficient condition such that these equations have periodic solutions. At the same time, the structure of the periodic solutions is also presented. Secondly, we characterize the existence and uniqueness of the periodic solutions in terms of the resolvent of the corresponding operators. As a matter of fact, our proof is quite elementary and natural, and gives a clear relationship between periodic solutions and the inhomogeneous term $f(t)$.

In Section 2, we begin with a functional differential equation on Hilbert space. As a result of Theorem 2.1, Theorem 2.2 extends the results in [4] and [6] to abstract space with improved proof. In Section 3 and 4, we deal with first and second order differential equations on Hilbert space. Here we find similar conditions for the existence of periodic solutions. In Theorem 3.1 we extend Pruss's result [9]. Section 4 presents a necessary and sufficient condition for the existence and uniqueness of 1-periodic solutions of second order differential equations (see also [10] for related results).

Let us fix some notations. First, for the sake of simplicity, we set the period $T = 1$. In a Hilbert space E, let $L^2([0,1], E)$ be the space of all measurable functions $f : [0,1] \to E$ such that $\|f\|_2 = (\int_0^1 |f(t)|^2 dt)^{1/2}$ is finite. $L^2([0,1], E)$ is a Hilbert space with w.r.t. the norm $\|\cdot\|_2$. The space of bounded continuous functions $f : [0,1] \to E$ is denoted by $C([0,1], E)$ and its norm is $\|f\|_0 = \sup_{t \in [0,1]} |f(t)|$. We recall some basic properties of Fourier coefficients of functions on Banach and Hilbert spaces.

Theorem 1.1 *Let E be a Banach space and let $f(t)$ and $g(t)$ be functions on $C([0,1], E)$. Then $f = g$ if and only if $f_n = g_n$ for all $n \in \mathbb{Z}$.*

Theorem 1.2 *Let E be a Banach space and $f \in C^1([0,1], E)$, i.e. f be continuously differentiable. Then $f_N(t) = \sum_{k=-N}^{N} f_k . e^{2k\pi it}$ converges uniformly to $f(t)$.*

Theorem 1.3 *Let E be a Hilbert space and $f \in C([0,1], E)$. Then the fol-*

lowing holds (Parseval equality).

$$\|f\|_2^2 = \int_0^1 |f(s)|^2 ds = \sum_{k\in Z} |f_k|^2.$$

For more details of Fourier series, we refer the readers to [2].

2. PERIODIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

On a Banach space E, consider the following functional differential equation

$$x'(t) = \int_0^\infty [dE(s)]x(t-s) + f(t), \tag{2.1}$$

where $E(s)$ is continuous from the left and is of bounded total variation on $[0,\infty)$, i.e.

$$0 \square \gamma = \int_0^\infty |dE(s)| < \infty. \tag{2.2}$$

Equation (1.1) has been studied by many authors on $\mathbb{C}^n$ (see [4], [6] and references therein). In this section we start with a new approach. For each function $f \in C([0,1],E)$ we give a necessary and sufficient condition for the existence of periodic solutions of (2.1) which is completely based on the Fourier series. The proof is natural and gives a new framework for abstract spaces. The idea comes from the observation that $x(t) = ae^{2n\pi it}$ is a solution of (2.1) corresponding to $f(t) = be^{2n\pi it}$ if and only if $(2n\pi i - A_n)a = b$, where A_n is defined on E by

$$A_n x = \int_0^\infty dE(s)e^{-2n\pi is}x. \tag{2.3}$$

This leads to the following

Theorem 2.1 *Let E be a Banach space and $\{E(t)\}_{t\geq 0}$ be a family of bounded operators in E satisfying* (2.2) *and $f \in C(E)$ be 1-periodic. Then the following are equivalent.*

(a) Equation (2.1) *has 1-periodic solutions.*

(b) *For each $n \in \mathbb{Z}$, $f_n \in Range(2\pi ni - A_n)$, where A_n is defined by (2.3), and there exists a sequence $\{x_n\}$, where $(2\pi in - A_n)x_n = f_n$, such that $\sum\limits_{-\infty}^{\infty} \|x_n\|^2 < \infty$ and $s_N = \sum\limits_{-N}^{N} x_n$ converges in E.*

Proof. $(a) \Rightarrow (b)$: Let $x(t)$ be a 1-periodic solution corresponding to f. Then we have the following relation:

$$(2n\pi i - A_n)x_n = f_n,$$

where f_n and x_n are Fourier coefficients of f and x, respectively. It implies $f_n \in Range(A_n)$ for $n \in \mathbb{Z}$. Moreover, by Theorem 1.2, the function $x_N(t) = \sum\limits_{n=-N}^{N} x_n e^{2\pi int}$ converges to $x(t)$ in $C([0,1],E)$. Hence, $\sum\limits_{-\infty}^{\infty} \|x_n\|^2 = \int_0^1 \|x(t)\|^2 dt < \infty$ and $s_N = \sum\limits_{-N}^{N} x_n = x_N(0)$ converges to $x(0)$ in E.

$(b) \Rightarrow (a)$: Let $\{x_n\}$ be a sequence satisfying $(2n\pi i - A_n)x_n = f_n$ and $s_N = \sum\limits_{-N}^{N} x_n$ converges in E. Define $f_N(t) = \sum_{n=-N}^{N} f_n e^{2\pi int}$ and $x_N(t) = \sum_{n=-N}^{N} x_n e^{2\pi int}$, then $x_N(t) \in C^1([0,1],E)$ is 1-periodic solution of (2.1) corresponding to $f_N(t)$. Integrating (2.1) from 0 to t we have

$$x_N(t) = x_N(0) + \int_0^t (\int_0^\infty dE(s)x_N(\tau - s))d\tau + \int_0^t f_N(\tau)d\tau. \qquad (2.4)$$

for $t \in [0,1]$. By Theorem 1.3, $f_N(t) \to f(t)$ and $x_N(t) \to x(t) = \sum_{n=-\infty}^{\infty} x_n e^{2\pi int}$ in $L^2([0,1],E)$ as $N \to \infty$. Hence, for $N > M$ we have

$$\begin{aligned} \|x_N(t) - x_M(t)\| \quad &\Box \quad \|x_N(0) - x_M(0)\| + \int_0^\infty |dE(s)| \int_0^1 \|x_N(\tau) - x_M(\tau)\|d\tau \\ &\qquad + \int_0^1 \|f_N(\tau - f_M(\tau)\|d\tau \\ &\Box \quad \|\sum_{M+1}^{N} x_i + \sum_{-N}^{-(M+1)} x_i\| + \int_0^\infty |dE(s)| \int_0^1 \|x_N(\tau) - x_M(\tau)\|^2 d\tau \\ &\qquad + \int_0^1 \|f_N(\tau - f_M(\tau)\|^2 d\tau \to 0 \end{aligned}$$

uniformly for $t \in [0,1]$. Hence $x_N(t) \to x(t)$ in $C([0,1],E)$. In particular, $x(t)$ is a 1-periodic function, and, by (2.4), is a solution of (2.1). $\Box$

From the above proof, it follows that all 1-periodic solutions of (2.1) are of the form $x(t) = \sum_{-\infty}^{\infty} x_n e^{2n\pi it}$, where x_n satisfy $(2n\pi i - A_n)x_n = f_n$ and the series converges uniformly. It is easy to see that if $(2n\pi i - A_n)$ is injective for all $n \in \mathbb{Z}$ then there is only one solution. The conversity is also true, since if $(2n\pi i - A_n)$ is not injective for some n, i.e, if there exists a $\tilde{x}_n \neq x_n$ such that $(2n\pi i - A_n)\tilde{x}_n = f_n$, then $\tilde{x}(t) = x(t) + \tilde{x}_n.e^{2n\pi it}$ is another 1-periodic solution. Using this observation, we are now in a position to characterize the existence and uniqueness of 1-periodic solutions of (2.1) for each $f(t) \in C([0,1], E)$.

Theorem 2.2 *Let E be a Hilbert space. Then the following are equivalent*

(a) For each continuous 1-periodic function f equation (2.1) *has one and only one continuous 1-periodic solution*

(b) $2n\pi i \in \rho(A_n)$ for all $n \in \mathbb{Z}$.

Proof $(a) \Rightarrow (b)$: Let (a) be satisfied. By the above observation, we only have to show that that $(2n\pi i - A_n)$ is surjective. Let y be an arbitrary point on E and $x(t)$ be the 1-periodic solution of $x'(t) = \int_0^\infty [dE(s)]x(t-s) + e^{2n\pi it}y$. If x_k is the k^{th} Fourier coefficient of $x(t)$, then $x_k = 0$ for all $k \neq n$ and $(2n\pi i - A_n)x_n = y$. Hence A_n is surjective.

$(b) \Rightarrow (a)$: By assumption, $2n\pi i \in \rho(A_n)$. Since $\|A_n\| \square \gamma$, there exists a constant C such that $\|(2n\pi i - A_n)^{-1}\| \square C/n$ for all $n \neq 0$. Let f be any function in $C([0,1], E)$. To show the existence of 1-periodic solution, by virtue of Theorem 2.1 it suffices to show that $s_N = \sum_{-N}^{N} (2k\pi i - A_k)^{-1} f_k$ converges in E. But for $N > M > 0$ we have

$$\begin{aligned} \|s_N - s_M\| &\square \sum_{M+1}^{N} \|(2k\pi i - A_k)^{-1} f_k\| + \sum_{-N}^{-(M+1)} \|(2k\pi i - A_k)^{-1} f_k\| \\ &\square \Big(\sum_{M+1}^{N} \|(2k\pi i - A_k)^{-1}\|^2\Big)^{1/2} \Big(\sum_{M+1}^{N} \|f_k\|^2\Big)^{1/2} \\ &\quad + \Big(\sum_{-N}^{-(M+1)} \|(2k\pi i - A_k)^{-1}\|^2\Big)^{1/2} \Big(\sum_{-N}^{-(M+1)} \|f_k\|^2\Big)^{1/2} \to 0 \end{aligned}$$

as $N, M \to \infty$. Thus s_n converges in E. The uniqueness of the solution follows from the injectivity of $(2n\pi i - A_n)$, and this completes the proof. □

3. PERIODIC SOLUTIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

On a Banach space E, we consider the first order differential equation

$$x'(t) = Ax(t) + f(t), \tag{3.1}$$

where A is the generator of a C_0-semigroup $(T(t))$ in E. A function $x \in C(\mathbb{R}, E)$ is called a mild solution of (3.1) if

$$x(t) = T(t-s)x(s) + \int_s^t T(t-\tau)f(\tau)ds \tag{3.2}$$

for all $t > s$ in $\mathbb{R}$ (see more details in [8]). If $x(t) \in C^1$, then it is called a strict solution. If $x(t)$ satisfies (3.2) on [0,1], such that $x(0) = x(1)$, then it is clear that $x(t)$ can be continuously extended to a 1-periodic mild solution of (3.1), provided f has been extended 1-periodically, too. Therefore, we call a mild solution of (3.1) *1-periodic* if it satisfies (3.2) for $0 \le s < t \le 1$ and $x(0) = x(1)$. We have the following.

Theorem 3.1 *Let A be the generator of a C_0-semigroup on a Hilbert space E and f be a continuous function on $[0,1]$. Then the following are equivalent.*

(a) Equation (3.1) *has 1-periodic mild solutions.*

(b) $f_n \in Range(2n\pi i - A)$ for each $n \in \mathbb{Z}$ and

$$\inf\{\sum_{n=-\infty}^{\infty} \|x_n\|^2 : \ (2\pi i n - A)x_n = f_n\} = M < \infty. \tag{3.3}$$

Proof. $(a) \Rightarrow (b)$: Let $x(t)$ be the 1-periodic solution corresponding to f. Then the Fourier coefficients of f and x satisfy the following equation (see [5]).

$$(2n\pi i - A)x_n = f_n. \tag{1}$$

It implies that $f_n \in Range(2n\pi i - A)$. Moreover, condition (3.3) is also satisfied since

$$\sum_{n=-\infty}^{\infty} \|x_n\|^2 = \int_0^1 \|x(t)\|^2 dt < \infty.$$

$(b) \Rightarrow (a)$: We borrow the technique obtained in [9]. Let $\{x_n\}$ be a sequence satisfying $(2n\pi i - A)x_n = f_n$ and $\sum_{n=-\infty}^{\infty} \|x_n\|^2 < \infty$. (Condition (3.3) guarantees the existence of this sequence). Define $f_N(t) = \sum\limits_{n=-N}^{N} f_n e^{2\pi int}$, $x_N(t) = \sum\limits_{n=-N}^{N} x_n e^{2\pi int}$. Then $f_N(t)$,and by assumption, $x_N(t)$, converge to $f(t)$ and a function $x(t)$ in $L^2([0,1],E)$ respectively. Moreover, $x_N(t) \in C^1[0,1],E)$ and is a 1-periodic solution of (3.1). Hence it satisfies

$$x_N(t) = T(t)x_N(0) + \int_0^t T(t-s)f_N(s)ds. \tag{3.4}$$

Taking $t = 1$ in (3.4), we get $(I - T(1))x_N(0) = \int_0^1 T(1-s)f_N(s)ds \to \int_0^1 T(1-s)f(s)ds$. On the other hand, multiplying (3.4) by $T(1-t)$ and integrating over $[0,1]$, we obtain

$$T(1)x_N(0) = \int_0^1 T(1-t)x_N(t)dt - \int_0^1 (\int_0^t T(1-\tau)f_N(\tau)d\tau)dt.$$

The right-hand side of this equation converges as $N \to \infty$. Hence $x_N(0) = T(1)x_N(0) + (1-T(1))x_N(0)$ converges in E. Therefore from (3.4) it follows that $x_N(t) \to x(t)$ in $C([0,1],E)$. In particular, $x(t)$ is a mild 1-periodic solution of (3.1). □

Again, all 1-periodic solutions of (3.1) have the structure $x(t) = \sum\limits_{-\infty}^{\infty}(2n\pi i - A)^{-1} f_n e^{2n\pi it}$. It is easy to see that these 1-periodic solutions are unique if and only if $(2n\pi i - A)$ is injective for each $n \in \mathbb{Z}$. In particular, if for every $f \in C[0,1]$, there exists a unique periodic solution of (3.1), then we get $2k\pi i \in \rho(A)$ for $k \in \mathbb{Z}$. Moreover,

$$\sum_{n=-\infty}^{\infty} \|(2n\pi i - A)^{-1} f_n\|^2 = \sum_{n=-\infty}^{\infty} \|x\|_n^2 < \infty$$

for each $f \in C([0,1], E)$. Using the similar argument as in Theorem 4.2, we obtain $sup_{k \in \mathbb{Z}} \|(2k\pi i - A)^{-1}\| < \infty$. Thus, the following Theorem, which was obtained in [9], is followed from the above observation.

Theorem *Let E be a Hilbert space and A be the generator of a C_0-semigroup on E. Then the following are equivalent.*

(a) For each continuous 1-periodic function f equation (3.1) *has one and only one 1-periodic mild solution*

(b) $2k\pi i \in \rho(A)$, $(k \in \mathbb{Z})$, and $sup_{k \in \mathbb{Z}} \|(2k\pi i - A)^{-1}\| < \infty$.

4. PERIODIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS

We are now consider the second order differential equation

$$u''(t) = Au(t) + f(t), \tag{4.1}$$

where A is the generator of a cosine family $(C(t))$ on a Banach space E. For a given function $f \in C(E)$, a function $u(t) \in C^1(E)$ is called a mild solution of (4.1) if

$$u(t) = C(t-s)u(s) + S(t-s)u'(s) + \int_s^t S(t-\tau)f(\tau)ds, \tag{4.2}$$

where $S(t)$ is the associated sine family (see [3] for more details). We call the mild solution $u(t)$ a strict solution if $u(t) \in C^2(\mathbb{R}, E)$. If $u(t)$ is a function satisfying (4.2) on [0,1] such that $u(0) = u(1)$ and $u'(0) = u'(1)$, then it is clear that $u(t)$ can be continuously extended by periodicity to a 1-periodic mild solution of (4.1), provided $f(t)$ has been extended 1-periodically, too. Therefore, we call a mild solution of (4.1) *1-periodic* if it satisfies (4.2) for $0 \leq s < t \leq 1$, $u(0) = u(1)$ and $u'(0) = u'(1)$. We have the following.

Theorem 4.1 *Let E be a Hilbert space and $f(t)$ be a function on $C([0,1], E)$. Then the following are equivalent.*

(a) Equation (4.1) *has 1-periodic mild solutions.*

(b) $f_n \in Range(4\pi^2 n^2 + A)$ *for* $n \in \mathbb{Z}$*, and*

$$\inf\{\sum_{-\infty}^{\infty} n^2 \|x_n\|^2 : (4\pi^2 n^2 + A)x_n = -f_n\} < \infty. \tag{4.3}$$

Proof. $(a) \Rightarrow (b)$: If $u(t)$ is a 1-periodic mild solution corresponding to $f(t)$, then the coefficients of f and u satisfy the following equation (see [7])

$$(4\pi^2 n^2 + A)u_n = -f_n.$$

It implies $f_n \in Range(4\pi^2 n^2 + A)$. Moreover, since $u(t)$ is in C^1 class, we have $u'(t) \in C([0,1], E)$ and the Fourier coefficients of $u'(t)$ are $u'_n = 2\pi niu_n$. Hence,

$$\sum_{n=-\infty}^{\infty} 4n^2\pi^2 \|u_n\|^2 = \int_0^1 \|u'(t)\|^2 dt < \infty,$$

from which (4.3) follows.

$(b) \Rightarrow (a)$: Let $\{x_n\}$ be a sequence satisfying $(4n^2\pi^2 + A)x_n = -f_n$ and $\sum_{n=-\infty}^{\infty} 4n^2\pi^2\|u_n\|^2 < \infty$. (Condition (4.3) guarantees the existence of such a sequence). Define $f_N(t) = \sum_{n=-N}^{N} f_n e^{2\pi int}$, $x_N(t) = \sum_{n=-N}^{N} x_n e^{2\pi int}$, where f_n is the Fourier coefficient of f and u_n satisfies $(4n^2\pi^2 - A)u_n = -f_n$. Then $f_N(t)$, and by assumption, $u_N(t)$ and $u'_N(t)$, converge to $f(t)$ and to certain functions $u(t)$ and $v(t)$ in $L^2([0,1], E)$, respectively. Moreover, for $N > M$ we have

$$\begin{aligned} \|u_N(t) - u_M(t)\| &\Box \sum_{M+1}^{N} \|u_n\| + \sum_{-N}^{-(M+1)} \|u_n\| \\ &\Box (\sum_{M+1}^{N} \frac{1}{n^2})^{1/2} (\sum_{M+1}^{N} n^2 \|u_n\|^2)^{1/2} \\ &\quad + (\sum_{-N}^{-(M+1)} \frac{1}{n^2})^{1/2} (\sum_{-N}^{-(M+1)} n^2 \|u_n\|^2)^{1/2} \to 0 \end{aligned}$$

as $N, M \to \infty$. This implies that $u_N(t)$ converges to function $u(t)$ uniformly. In particular, u is a continuous and 1-periodic function. To prove that u is in

C^1 class, $u'(0) = u'(1)$, and satisfies (4.2), it suffices to show the convergence of $u'_N(t)$ in $C([0,1],E)$ as $N \to \infty$. To do that, we first observe that $u_N(t)$ is a 1-periodic mild solution of (4.1) corresponding to $f_N(t)$, i.e.,

$$x_N(t) = C(t)x_N(0) + S(t)u'_N(0) + \int_0^t S(t-s)f_N(s)ds. \qquad (4.4)$$

From (4.4) it follows

$$u'_N(t) = C'(t-s)u_N(s) + C(t-s)u'_N(s) + \int_s^t C(t-\tau)f_N(\tau)d\tau. \qquad (4.5)$$

Multiplying (4.4) by $C'(t-s)$, (4.5) by $C(t-s)$, and subtracting the results, we have

$$C(t-s)u'_N(t) - C'(t-s)u_N(t) = u'_N(s) + \int_s^t C(\tau - s)f_N(\tau)d\tau. \qquad (4.6)$$

Integrating (4.6) over $[s, s+1]$ and using the fact that $\int_s^{s+1} C'(t-s)u_N(t)dt = (C(1) - I)u_N(s) - \int_s^{s+1} C(t-s)u'_N(t)dt$, we conclude that

$$2\int_s^{s+1} C(t-s)u'_N(t)dt - (C(1)-I)u_N(s) =$$

$$. \qquad u'_N(s) + \int_s^{s+1}\Big(\int_s^t C(\tau - s)f_N(\tau)d\tau\Big)dt \text{ for all } s \in [0,1]. \qquad (4.7)$$

By assumption, $\sum\limits_{-\infty}^{\infty} 4\pi^2 n^2 \|u_n\|^2 < \infty$. It implies that $u'_N(t)$ converges in $L^2([0,1],E)$. Hence, from (4.7) it follows the convergence of $u'_N(s)$ in $C([0,1],E)$ and it completes the proof. □

It is interesting that the periodic solutions of (4.1) are also of the form $u(t) = \sum\limits_{\infty}^{\infty} u_n e^{2n\pi it}$, where $(4n^2\pi^2 + A)u_n = -f_n$. Moreover, if condition(4.3) is satisfied, then $u(t)$ automatically belongs to C^1 class. We now characterize the existence and uniqueness of periodic solution of second order in terms of the spectrum of A.

Theorem 4.2 *Let E be a Hilbert space. Then the following are equivalent.*

(a) Equation (4.1) *has a unique 1-periodic mild solution for each function $f \in C([0,1],E)$.*

(b) $\{-4\pi^2 n^2 : n \in \mathbb{Z}\} \subset \rho(A)$ *and there is a constant* C *such that for all* $n \neq 0$, $\|(4\pi^2 n^2 + A)^{-1}\| < C/n$.

Proof. $(a) \to (b)$ follows directly from Theorem 4.1. To prove the inverse, let $f(t)$ be any function in $C([0,1], E)$ and $x(t)$ be the corresponding 1-periodic solution. Using the equation

$$(4\pi^2 n^2 + A)u_n = -f_n \text{ for every } n \in \mathbb{Z},$$

and the arguments as in the proof of Theorem 2.2, we obtain that $(4\pi^2 n^2 + A)$ is surjective and injective, i. e.,$-4\pi^2 n^2 \in \rho(A)$ and $u_n = -(4\pi^2 n^2 + A)^{-1} f_n$.

Now let u'_n be the Fourier coefficients of u', then $u'_n = 2\pi n i u_n$. Hence,

$$\sum_{-\infty}^{\infty} 4\pi^2 n^2 \|(4\pi^2 n^2 + A)^{-1} f_n\|^2 = \sum_{-\infty}^{\infty} \|u'_n\|^2 = \int_0^1 \|u'(t)\|^2 dt < \infty \qquad (4.8)$$

for every $f \in C([0,1], E)$. Assume contrarily that (b) is not true, i.e. $sup_n \|n(4\pi^2 n^2 + A)^{-1}\| = \infty$. Then we can find a sequence $(n_j)_{j \in \mathbb{N}}$ with $|n_j| \to \infty$ and $\|n_j(4\pi^2 n_j^2 - A)^{-1}\| \to \infty$. Hence, from sequence $(n_j)_j$ we can find a subsequence, and without loss of generality, denoted again by $(n_j)_j$, such that for each n_j there is a point f_{n_j} with $\|f_{n_j}\| = 1$ and $\|n_j(4\pi^2 n_j^2 - A)^{-1} f_{n_j}\| > j^2$.

Let $g_{n_j} = \frac{1}{j^2} f_{n_j}$, then $\|g_{n_j}\| = \frac{1}{j^2}$ and $\|n_j((4\pi^2 n_j^2 - A)^{-1} g_{n_j}\| > 1$. It follows that $\sum_{j=0}^{\infty} g_{n_j} e^{-2\pi j i t}$ uniformly converges in $C([0,1], E)$. Hence, $g = \sum_{j=0}^{\infty} g_{k_j} e^{-2\pi j i t}$ is a function in $C([0,1], E)$, where

$$\sum_{j=0}^{\infty} \|n_j(4\pi^2 n_j^2 - A)^{-1} g_{n_j}\|^2 = \infty,$$

which is contradictory to (4.8), and this completes the proof. □.

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